Polarization dynamics of solitons in birefringent fibers

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(Received 18 October 1999)

We study the dynamics of uniformly polarized pulses in a birefringent optical fiber. By considering the Hamiltonian structure, symmetries, and the momentum map of the underlying equations, we obtain a self-consistent set of equations for the polarization state alone. In the autonomous case, we find the bifurcation curve of this system, and discuss how the orbits change in the neighborhood of this curve. We calculate the orbits explicitly. An extension to nonautonomous underlying equations is also possible. We further briefly discuss the effect of radiation emission from solitons as their polarization state changes.

PACS number(s): 42.25.Lc, 42.25.Ja

I. EQUATIONS OF MOTION

The propagation of an optical pulse down a birefringent optical fiber is usefully described by the perturbed vector nonlinear Schrödinger equation [1–3]:

$$i\frac{\partial \mathbf{q}}{\partial z}-\frac{\partial^2 \mathbf{q}}{\partial t^2}-2\mathbf{q}(\mathbf{q}^\dagger\mathbf{q}) +\beta\sigma_1\mathbf{q}-i\beta'\sigma_1\frac{\partial \mathbf{q}}{\partial t}-\gamma(z)\sigma_3\mathbf{q}+2B\sigma_3\mathbf{q}(\mathbf{q}^\dagger\sigma_3\mathbf{q})=\mathbf{0}. \tag{1}$$

Here $\mathbf{q}=(q_1,q_2)^T$, where q_1,q_2 are the components of the envelope pulse in each of the linearly polarized modes; q_1 and q_2 are the amplitudes of the *fast* and *slow* modes, respectively. The independent variable z is the distance of propagation down the fiber, and t is a retarded time variable. The term $(\partial^2\mathbf{q}/\partial t^2)$ is the usual second order dispersion term. The symbol † denotes the Hermitian conjugate, so that

$$\mathbf{q}\mathbf{q}^\dagger\mathbf{q}=(|q_1|^2+|q_2|^2)\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{2}$$

The parameters β and β', which are both taken to be positive, are respectively the weak and strong birefringence parameters, so that β is proportional to the difference in phase velocity between the two modes, while β' is proportional to the difference of their group velocities. These two quantities are frequency dependent; β' is the derivative of β. Here they are evaluated at the carrier frequency, and treated as constants.

The parameter $\gamma(z)$ allows for twisting of the fiber axes with distance down the fiber. It is often presumed that γ varies randomly with distance down the fiber, in accordance with properties of ''real'' fibers [4,5]. It has been pointed out that, by a change of variables, one may remove the term $\gamma(z)\sigma_3\mathbf{q}$; however, this is at the cost of replacing σ_1 in the remaining polarization-dependent terms by a z-dependent matrix. Of course, the underlying dynamics will be independent of the coordinates used.

Finally, if the last term in Eq. (1) is expanded, it assumes the more recognizable form:

$$2B\begin{bmatrix} |q_2|^2q_1-q_2^2q_1^* \\ |q_1|^2q_2-q_1^2q_2^* \end{bmatrix}, \tag{3}$$

where * denotes the complex conjugate. For optical fibers, B equals 1/3, although in many studies it is treated as a suitable small parameter, to permit study of Eq. (1) using perturbation theory [6].

Here and throughout this paper, the σ_j are the (renumbered) Pauli matrices:

$$\sigma_1=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad \sigma_2=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_3=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{4}$$

In an early study of nonlinear interaction in waveguides, Wabnitz and co-workers [7–9] investigated Eq. (1) in the limit of cw beams, so that all terms with time derivative $\partial/\partial t$ are identically zero. Using the variables

$$S_j=\frac{\mathbf{q}^\dagger\sigma_j\mathbf{q}}{\mathbf{q}^\dagger\mathbf{q}},\quad j=1,2,3, \tag{5}$$

they obtained a ''torque'' equation for the evolution of $\mathbf{S}=(S_1,S_2,S_3)$ in the form

$$\frac{d\mathbf{S}}{dz}=\mathbf{S}\times\mathbf{\Omega}. \tag{6}$$

The vector $\mathbf{S}$ is the Stokes vector; this is a unit vector that characterizes the polarization state of the beam completely. Its significance is discussed later. Any point on the unit sphere characterized by the coordinates (S_1,S_2,S_3) corresponds to a unique polarization state of the electromagnetic field. The equator $S_3=0$ corresponds to linearly polarized beams, while the two poles $S_3=\pm 1$ correspond to the two circularly polarized states. Wabnitz *et al.* analyzed the bifurcations that occur as the parameters β, γ, and B are varied. Their analysis is extended here, first by showing that the same torque equation also describes the polarization dynamics of soliton pulses, and by deriving its Hamiltonian structure, then by deriving the bifurcation curve, an astroid in the $\beta-\gamma$ plane, which completely distinguishes all the different types of evolution of the polarization state.

A central feature of this paper is the introduction of the momentum map; this enables us to reduce Eq. (1) to a specific Lie-Poisson form of the torque equation not previously reported in this context; this is

$$\frac{d\mathbf{S}}{dz}=\mathbf{S}\times\frac{\partial H}{\partial \mathbf{S}}. \tag{7}$$

Here $H(\mathbf{S})$ is the reduced Hamiltonian function, obtained from Eq. (1), which is determined by the properties of the host medium and the pulse envelope; it describes the dependence of the energy on the polarization state. This function thus determines the manner in which the polarization state evolves. For example, in the simplest case of Faraday rotation, where the polarization eigenstates are right and left circularly polarized light, $H=\beta S_3$, where β is the difference between the phase velocities of the two modes. In general, the eigenstates correspond to points on the sphere where $\mathbf{S}\times(\partial H/\partial \mathbf{S})=\mathbf{0}$ so that $\mathbf{S}$ is either parallel or antiparallel to $\partial H/\partial \mathbf{S}$. In this case, the eigenmodes are seen to be the poles, $S_3=\pm 1$, where it can be seen that H takes its maximum and minimum values on the sphere. If the initial polarization state of the field is linear, Eq. (7) indicates that it will remain linearly polarized, but that the polarization axis will rotate at the constant rate β. This description of Faraday rotation is of course well known—what we emphasize here is its description in terms of the Hamiltonian system (7). The appropriate form of $H(\mathbf{S})$ corresponding to Eq. (1) is derived below. In this case, the Hamiltonian formalism enables us to give a straightforward geometrical description of the orbits of the reduced system, and their bifurcations. While these could be derived from the equations directly, the Lie-Poisson approach used here is more natural and, we believe, instructive, when discussing a Hamiltonian system with symmetry, such as Eq. (1).

The first treatment of the evolution of the polarization state of a soliton pulse was reported by Akhmediev and others in a series of papers [11–13]. Here $\mathbf{S}$ is now an averaged polarization state for the soliton, as will be described below, and the resulting evolution equation for $\mathbf{S}(z)$ is again the torque equation given by Wabnitz *et al.* A discussion of possible bifurcations and types of evolution of $\mathbf{S}(z)$ is reported in [11,12] for the symmetric case when the parameter γ in Eq. (1) is set to zero. In [13], the approach is applied to a slightly different system which, however, has similar symmetry properties. Here we extend their approach in three ways: first by using the momentum map to show how Eq. (7) can be derived systematically from the Hamiltonian functional $\mathcal{H}(\mathbf{q})$, which generates the perturbed nonlinear Schrödinger equation (1), second by extending to the significantly more complex asymmetric case, and third by discussing the effects of a nonautonomous perturbation such as the z-dependent γ discussed above.

II. MOMENTUM MAP

Let us consider the unperturbed form of Eq. (1), with $\beta=\beta'=\gamma=B=0$:

$$i\mathbf{q}_z-\mathbf{q}_{tt}-2\mathbf{q}(\mathbf{q}^\dagger\mathbf{q})=\mathbf{0}. \tag{8}$$

This is a completely integrable system; Manakov gave its zero curvature representation in [14]. More relevant here is its invariance under the action of the Lie group SU(2), for if

$$\tilde{\mathbf{q}}=U\mathbf{q},\qquad \tilde{\mathbf{q}}^\dagger=\mathbf{q}^\dagger U^\dagger, \tag{9}$$

and U is a 2×2 unitary matrix, then $\tilde{\mathbf{q}}$ will also satisfy Eq. (8). The derivative of this action at the identity is an action of the Lie algebra su(2):

$$X_{\mathbf{A}}(\mathbf{q})=\mathbf{A}\mathbf{q},\qquad X_{\mathbf{A}}(\mathbf{q}^\dagger)=\mathbf{q}^\dagger\mathbf{A}^\dagger. \tag{10}$$

Here $\mathbf{A}$ can be any traceless skew-Hermitian matrix; these can be expanded in any appropriate basis, for instance,

$$\mathbf{A}=i\sum_{j=1}^{3} a_j\sigma_j, \tag{11}$$

where the σ_j are the Pauli matrices defined as in Eq. (4) above. Their commutation relations are

$$[i\sigma_j,i\sigma_k]=-2i\epsilon_{jkl}\sigma_l. \tag{12}$$

The coefficients a_j are real. Now both Eqs. (8) and (10), as well as the perturbed Eq. (1) have the same canonical Hamiltonian structure, with the Poisson bracket given by

$$\{H,J\}=i\int_{-\infty}^{\infty}\frac{\delta H}{\delta\mathbf{q}}\frac{\delta J}{\delta\mathbf{q}^\dagger}-\frac{\delta J}{\delta\mathbf{q}}\frac{\delta H}{\delta\mathbf{q}^\dagger}dt. \tag{13}$$

The Hamiltonian for Eq. (8) is

$$H=\int_{-\infty}^{\infty}\mathbf{q}_t^\dagger\mathbf{q}_t-(\mathbf{q}^\dagger\mathbf{q})^2dt, \tag{14}$$

while that for the vector field $X_{\mathbf{A}}$ is

$$J(\mathbf{A})=\int_{-\infty}^{\infty}-i(\mathbf{q}^\dagger\mathbf{A}\mathbf{q})dt=\mathrm{Tr}\left(-i\mathbf{A}\int_{-\infty}^{\infty}\mathbf{q}\mathbf{q}^\dagger dt\right)=\mathrm{Tr}\left(-i\mathbf{A}\int_{-\infty}^{\infty}(\mathbf{q}\mathbf{q}^\dagger-\mathbf{I}\mathbf{q}^\dagger\mathbf{q}/2)dt\right)=\mathrm{Tr}(-i\mathbf{A}\mathbf{J}). \tag{15}$$

Here $\mathbf{I}$ is the 2×2 identity matrix. If the matrix $\mathbf{A}$ is considered as an element of the Lie algebra SU(2), then the expression, known as the momentum map (see, for instance, [15]),

$$\mathbf{J}=\int_{-\infty}^{\infty}(\mathbf{q}\mathbf{q}^\dagger-\mathbf{I}\mathbf{q}^\dagger\mathbf{q}/2)dt \tag{16}$$

should be understood as taking values in its dual, SU(2)*. We can expand **J**, which is traceless and Hermitian, in the dual basis:

$$\mathbf{J}=\frac{1}{2}\sum_{j=1}^{3}J_j\sigma_j,$$
$$J_j=J(\sigma_j)=\mathrm{Tr}(\sigma_j\mathbf{J})=\tfrac{1}{2}\int_{-\infty}^{\infty}\mathbf{q}^\dagger\sigma_j\mathbf{q}dt. \tag{17}$$

The components J_j are real. Their Poisson brackets, given by Eq. (13), have identical structure constants to the basis matrices $i\sigma_j$:

$$\{J_j,J_k\}=i\int_{-\infty}^{\infty}\frac{\delta J_j}{\delta\mathbf{q}}\frac{\delta J_k}{\delta\mathbf{q}^\dagger}-\frac{\delta J_k}{\delta\mathbf{q}}\frac{\delta J_j}{\delta\mathbf{q}^\dagger}dt=i\int_{-\infty}^{\infty}\mathbf{q}^\dagger[\sigma_j,\sigma_k]\mathbf{q}dt$$
$$=-2\int_{-\infty}^{\infty}\epsilon_{jkl}\mathbf{q}^\dagger\sigma_l\mathbf{q}dt=-2\epsilon_{jkl}J_l. \tag{18}$$

The components J_j are, up to a multiplicative factor, the Stokes parameters, which are the standard variables in which to discuss polarization phenomena in linear media. The action of the symmetry (9) on **J**, considered now as the three-component vector $(J_1,J_2,J_3)^T$, is an orthogonal rotation. Hence the orbit of **J** under the group SU(2) is a sphere; this is of course the Poincaré sphere.

We will see below that these variables give a natural and effective description of nonlinear polarization dynamics in birefringent media. First it is necessary to restrict our attention to states of the system for which the polarization is well defined.

III. CANONICAL EVOLUTION EQUATION

Now we consider the perturbed system (1). This can be written as

$$\mathbf{q}_z=\{\mathcal{H}+\mathcal{H}_I,\mathbf{q}\}=-i\frac{\delta(\mathcal{H}+\mathcal{H}_I)}{\delta\mathbf{q}^\dagger}, \tag{19}$$

where the Poisson bracket is defined as in Eq. (13), the Hamiltonian $\mathcal{H}$ is given above in Eq. (14), and the perturbations are generated by the Hamiltonian

$$\mathcal{H}_I=\int_{-\infty}^{\infty}\beta\mathbf{q}^\dagger\sigma_1\mathbf{q}-\frac{i\beta'}{2}(\mathbf{q}^\dagger\sigma_1\mathbf{q}_t-\mathbf{q}^\dagger{}_t\sigma_1\mathbf{q})-\gamma\mathbf{q}^\dagger\sigma_3\mathbf{q}$$
$$+B(\mathbf{q}^\dagger\sigma_3\mathbf{q})^2dt. \tag{20}$$

The first two conserved quantities for the unperturbed system are

$$c_0=\int_{-\infty}^{\infty}\mathbf{q}^\dagger\mathbf{q}dt, \tag{21}$$

$$c_1=\frac{i}{2}\int_{-\infty}^{\infty}(\mathbf{q}^\dagger\mathbf{q}_t-\mathbf{q}_t^\dagger\mathbf{q})dt. \tag{22}$$

These are also conserved quantities for the perturbed system (1). Using Eqs. (18) and (19), we find

$$\frac{dJ_j}{dz}=i\int_{-\infty}^{\infty}\left(\mathbf{q}^\dagger\sigma_j\frac{\delta\mathcal{H}_I}{\delta\mathbf{q}^\dagger}-\frac{\delta\mathcal{H}_I}{\delta\mathbf{q}}\sigma_j\mathbf{q}\right)dt. \tag{23}$$

The contribution to this from the first and third terms in Eq. (20) is easily found to be

$$\epsilon_{ijk}J_j\frac{\partial}{\partial J_k}(\beta J_1-\gamma J_3). \tag{24}$$

It is not possible to express the other two terms this simply—these give

$$-\beta'\epsilon_{ij1}\int_{-\infty}^{\infty}(\mathbf{q}_t^\dagger\sigma_j\mathbf{q}-\mathbf{q}^\dagger\sigma_j\mathbf{q}_t)dt \tag{25}$$

and

$$-4iB\epsilon_{im3}\int_{-\infty}^{\infty}(\mathbf{q}^\dagger\sigma_m\mathbf{q})(\mathbf{q}^\dagger\sigma_3\mathbf{q})dt, \tag{26}$$

respectively. In this paper, however, we will suppose that the polarization state does not change along the length of the pulse, and that the pulse profile does not vary with z, which is the case for vector solitons. This property has been verified experimentally by Evangelides *et al.* [16]. Then we have $\mathbf{q}(z,t)=q(t-v^{-1}z)e^{i\phi(z)}\mathbf{c}(z)$, where **c** is a unit vector (that is, satisfying $\mathbf{c}^\dagger\mathbf{c}=1$), and $q(z,t)$ is a scalar. The velocity v and phase ϕ are irrelevant to the polarization dynamics discussed here. The vector **c** carries complete information about the polarization state. Then we see that

$$J_i=c_0\mathbf{c}^\dagger\sigma_i\mathbf{c} \tag{27}$$

and the terms (25) and (26) become

$$i\epsilon_{ijk}J_j\frac{\partial}{\partial J_k}\left(2\frac{c_1}{c_0}\beta' J_1\right) \tag{28}$$

and

$$-i\epsilon_{ijk}J_j\frac{\partial}{\partial J_k}\left(2\frac{\alpha}{c_0}BJ_3^2\right), \tag{29}$$

where $\alpha=\int_{-\infty}^{\infty}(\mathbf{q}^\dagger\mathbf{q})^2dt/c_0$. This is a constant for the class of pulses considered. In particular, if the profile is the unperturbed soliton $q=\eta\,\mathrm{sech}(\eta t)$, then $\alpha=\frac{2}{3}\eta^2$. Thus the evolution equation for **J** may be written in the Lie-Poisson form

$$\frac{dJ_i}{dz}=\epsilon_{ijk}J_j\frac{\partial H}{\partial J_k}, \tag{30}$$

where the new Hamiltonian function H is

$$H=\left(\beta-2\frac{c_1}{c_0}\beta'\right)J_1-\gamma J_3+2\frac{B\alpha}{c_0}J_3^2. \tag{31}$$

This function is the restriction of the function $\mathcal{H}_I$ to uniformly polarized pulses. Finally, it is convenient to introduce the unit vector in the direction of **J**:

$$\mathbf{S}=\mathbf{J}/c_0. \tag{32}$$

We also note that $\beta-2c_1\beta'/c_0$ is the first-order Taylor expansion for the parameter $\beta(\omega)$, evaluated at the spectral peak of the soliton pulse ω_0-2c_1/c_0, rather than at the reference (carrier) frequency ω_0. We denote this expression hereafter as just β. Hence our final result is that

$$\frac{d\mathbf{S}}{dz}=\mathbf{S}\times\frac{\partial H}{\partial\mathbf{S}}, \qquad (33)$$

where

$$H=\beta S_1-\gamma S_3+2\alpha B S_3^2. \qquad (34)$$

In components, this is

$$\frac{d}{dz}\begin{pmatrix}S_1\\S_2\\S_3\end{pmatrix}=\begin{pmatrix}S_2(S_3-\gamma)\\-S_1(S_3-\gamma)+\beta S_3\\-\beta S_2\end{pmatrix}. \qquad (35)$$

IV. BIFURCATIONS OF THE DYNAMICS

We can now discuss the orbits of Eq. (34). By conservation of energy, these must be the intersections of level sets of H with the Poincaré sphere; that is, solutions of the two equations

$$\frac{1}{2}S_3^2-\overline{\gamma}S_3+\overline{\beta}S_1=e, \qquad (36)$$

$$S_1^2+S_2^2+S_3^2=1. \qquad (37)$$

Here, we have introduced $\overline{\gamma}=\gamma/(4\alpha B)$ and $\overline{\beta}=\beta/(4\alpha B)$, while the left hand side of Eq. (36) is the normalized Hamiltonian $\overline{H}=H/(4\alpha B)$. The Hamiltonian structure is unchanged if we then replace z by $\overline{z}=4z\alpha B$. These level sets are a three-parameter family of parabolic cylinders, labeled by the parameters $\overline{\beta}$, $\overline{\gamma}$, and the normalized energy e. We need to understand the singularities of this family. The simplest of these are the fixed points, where the sphere is tangent to one of the parabolic cylinders, $\overline{H}(\mathbf{S})=e$. From the equation of motion (34), we see that all fixed points lie in the plane $S_2=0$, which is clearly a plane of symmetry of both the sphere and $\overline{H}$. All periodic orbits must cross this plane, for otherwise S_3 would be monotonic. If we eliminate S_1 at these intersection points, we get a quartic polynomial in S_3:

$$Q(S_3)=-\left(\frac{1}{2}S_3^2-\overline{\gamma}S_3-e\right)^2+\overline{\beta}^2(1-S_3^2)=0. \qquad (38)$$

Now the number of real roots of this quartic must be even. Note that if $S_3^2>1$ and $\overline{\beta}$ is real, then $Q(S_3)<0$, so any real roots must occur in the interval $-1\leqslant S_3\leqslant 1$, corresponding to real points of the sphere. If there are no real roots, then the sphere and cylinder do not touch and the orbit is empty. If there are two roots, the orbit is a simple closed curve, crossing $S_2=0$ at these two points. If, however, all four roots are real, then the sphere must intersect the energy surface *twice* in two disjoint curves, which are distinct orbits of the system. For any $\overline{\beta},\overline{\gamma}$, there will be some values of e where two or more of these roots coincide. A double root corresponds to a fixed point of the motion, which is a point of tangency of the sphere and the energy surface. There must always be at least two of these, for H must achieve its maximum and minimum value on the sphere. Clearly any double root of $Q(S_3)$ is a common root of $Q(S_3)$ and the cubic $Q'(S_3)$. Eliminating S_3 from these simultaneous equations for S_3 and e gives another quartic equation, in the energy e, $\tilde{Q}(e)=0$. We will not write this out explicitly. As we have just seen, this must have at least two real roots, but depending on the values of $\overline{\beta}$ and $\overline{\gamma}$, the remaining two roots may be either complex or real. In the former case, the intersection of the sphere with *any* energy surface $\overline{H}=e$ is a simple closed curve—thus the Hamiltonian $\overline{H}$ has one maximum and one minimum on the sphere, but no other critical points. In the latter case, there are some energy surfaces that cut the sphere twice, giving two disjoint closed curves. This is illustrated in Fig. 1. Here $\overline{H}$ still has a maximum and a minimum, as well as two other critical points—these must be a saddle and a second maximum. The saddle will of course be an unstable fixed point of the motion. The homoclinic orbit through this saddle will have two lobes; these are separatrices bounding the three families of closed curves surrounding the other three critical points. This is illustrated in Fig. 2.

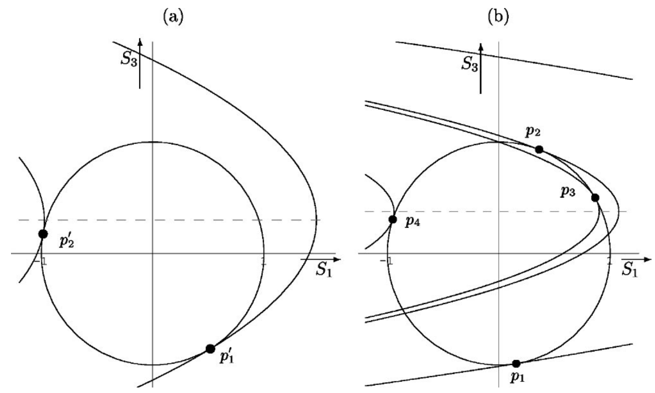

FIG. 1. Energy level surfaces corresponding to the roots of the quartic $\tilde{Q}(e)=0$. (a) $\tilde{Q}$ has only two real roots, $e_1'>e_2'$, the maximum and minimum of H on the sphere. (b) $\tilde{Q}$ has four real roots, $e_1>e_2>e_3>e_4$. Note that the level sets of maximum and minimum energy, e_1 and e_4, touch the sphere only at their points of tangency, p_1 and p_4, whereas the sets of energy e_2 and e_3 also have two transversal intersections, as well as the points of tangency p_2 and p_3.

Finally, we need to understand how the fixed points can bifurcate. There are two approaches here—we can look for double roots of $\tilde{Q}(e)$, or more simply, we can look for triple roots of $Q(S_3)$. If, at the triple root, we set $S_3=\cos(\Theta)$, and require simultaneously that

$$Q(S_3)=Q'(S_3)=Q''(S_3)=0, \qquad (39)$$

we get, for given Θ, three simultaneous equations for $(\overline{\beta},\overline{\gamma},e)$:

$$\frac{1}{2}\cos^2(\Theta)-\overline{\gamma}\cos(\Theta)+\overline{\beta}\sin\Theta-e=0,$$

$$-\sin(\Theta)\cos(\Theta)+\overline{\gamma}\sin(\Theta)+\overline{\beta}\cos\Theta=0, \qquad (40)$$

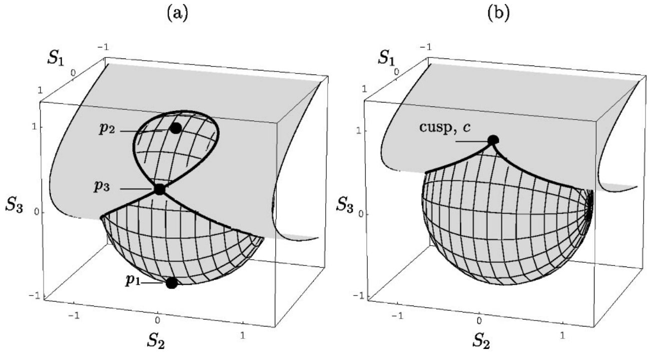

FIG. 2. (a) Intersection of level set $H(S)=e_3$ with the Poincaré sphere, showing the lobes of the homoclinic orbit through the saddle p_3. (b) As points p_2 and p_3 coalesce, one lobe shrinks to a point c, which is a cusp of the remaining lobe.

$$\sin^2(\Theta)-\cos^2(\Theta)+\overline{\gamma}\cos(\Theta)-\overline{\beta}\sin\Theta=0.$$

From here it is elementary to find

$$e=1-\frac{3}{2}\cos^2(\Theta),$$

$$\overline{\beta}=\sin^3(\Theta), \tag{41}$$

$$\overline{\gamma}=\cos^3(\Theta).$$

We see therefore that all these bifurcation points lie on the astroid

$$\overline{\beta}^{2/3}+\overline{\gamma}^{2/3}=1. \tag{42}$$

This is a closed curve in the $(\overline{\beta},\overline{\gamma})$ plane, with cusps at the points $(\pm1,0)$ and $(0,\pm1)$. It is illustrated in Fig. 3.

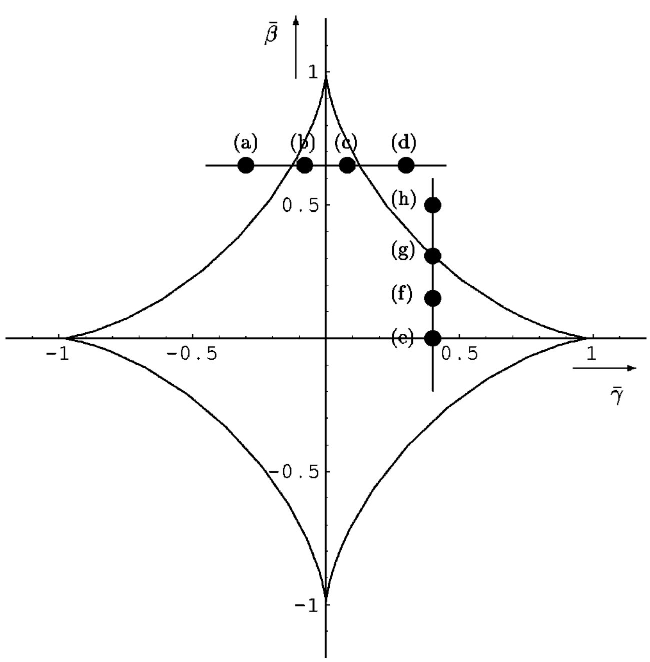

FIG. 3. The astroid $\overline{\beta}^{2/3}+\overline{\gamma}^{2/3}=1$. Some orbits, corresponding to points (a)–(h) are shown in Fig. 4.

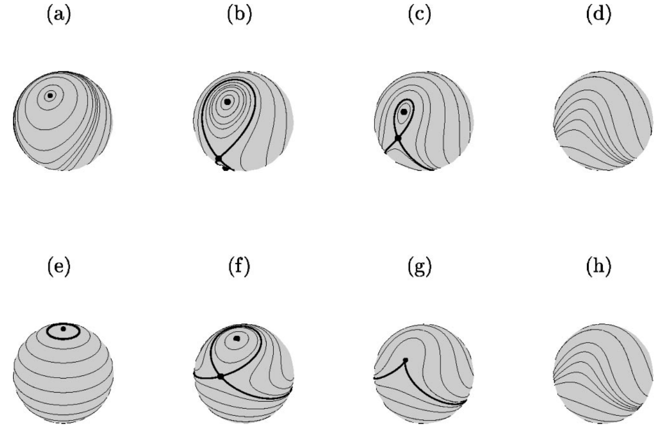

FIG. 4. The orbits corresponding to the points $(\overline{\beta},\overline{\gamma})$ shown in Fig. 3.

If $(\overline{\beta},\overline{\gamma})$ lies outside this curve, the linear terms in $\overline{H}$ predominate, $\overline{H}$ has two critical points, and the orbits are all simple closed curves. In this case we expect the motion to be a deformation of that found in the linear model, in which **S** precesses at constant rate around a circle on the sphere. If, however, $(\overline{\beta},\overline{\gamma})$ lies inside the curve, then the nonlinear term is more important, and $\overline{H}$ will have a saddle point on the sphere, and a homoclinic orbit through this point.

As the curve is approached, the saddle and one of the other critical points will coalesce. The lobe of the homoclinic orbit surrounding this extremum will then shrink onto the saddle point; the other lobe of the homoclinic orbit will then have a cusp at this point. This is shown in Fig. 2(b). Some other representative orbits are shown in Fig. 4.

There are further singularities at the cusps of the bifurcation curve; at these points $\overline{H}$ has higher symmetry than we have considered so far. There are two cases, on the $\overline{\gamma}$ and $\overline{\beta}$ axes. In the former case, when $\overline{\beta}=0$, $\overline{H}$ is invariant under rotations about the S_3 axis, so we can see at once that the orbits are small circles on the sphere; $S_3=\text{const}$. The Hamiltonian in this case is $\overline{H}=\frac{1}{2}S_3^2-\overline{\gamma}S_3$, so the rate of precession about the axis is

$$\omega_3=\frac{\partial\overline{H}}{\partial S_3}=S_3-\overline{\gamma}. \tag{43}$$

If $|\overline{\gamma}|>1$, then ω will not vanish on the sphere, and the only fixed points are at the poles $S_3=\pm1$. However, if $|\overline{\gamma}|<1$, then one circle $S_3=|\overline{\gamma}|$ will have vanishing rate of precession—it is a locus of fixed points. The rate of precession changes sign as this locus is crossed. Now if we let $\overline{\gamma}$ tend to ±1, this stationary locus will collapse onto the fixed point at the pole $S_3=\pm1$, which will be neutrally stable. This singularity is known as a *Hamiltonian Hopf bifurcation* [10].

The other special case occurs when $\overline{\gamma}=0$. This case was discussed by Akhmediev and Soto-Crespo [11]. Here $\overline{H}$ has the reflection symmetry $S_3\rightarrow-S_3$, and so the orbits, or pairs of them, must have this symmetry too. Here, if $\overline{\beta}>1$, as in the asymmetrical case, the orbit is a simple closed curve, while for $0<\overline{\beta}<1$, the point $S_1=1$ becomes a saddle, with a symmetrical pair of homoclinic orbits on either side of it. As $\overline{\beta}\to 1$, these orbits shrink onto the saddle, so that at $\overline{\beta}=1$, it becomes a neutrally stable fixed point. This is called a *Hamiltonian pitchfork bifurcation*. As $\overline{\beta}\to 0$ from above, the two lobes of the homoclinic orbit approach the equator $S_3=0$ from either side, so that it becomes, as we have seen, a locus of fixed points. For negative $\overline{\beta}$ the process is repeated in reverse, with the saddle point now being at $S_1=-1$.

In their analysis of the cw case, Matera and Wabnitz [9] introduced the parameters p and t, as measures of the normalized ''power'' in the cw beam, and the ''twist ratio'' of the fiber, respectively. These are effectively related to our parameters $\overline{\beta}$ and $\overline{\gamma}$ by $p=1/\overline{\beta}$, and $t=\overline{\gamma}/\overline{\beta}$; there, however, p depends on the power of the cw beam, rather than on the energy of the soliton pulse. The bifurcation curve (42) becomes, in these variables, $p^{2/3}-t^{2/3}=1$. The parameter α introduced above may be identified with p in the cw case.

V. ORBITS

For general $(\overline{\beta},\overline{\gamma})$, the equation of motion for S_3 reduces to the simple form

$$\left(\frac{dS_3}{d\overline{z}}\right)^2=Q(S_3), \tag{44}$$

which can be solved explicitly in terms of Weierstrass elliptic $\wp$ functions. Here we denote

$$Q(S_3)=a_0S_3^4+4a_1S_3^3+6a_2S_3^2+4a_3S_3+a_4, \tag{45}$$

so that

$$a_0=-1/4, \tag{46}$$

$$a_1=\overline{\gamma}/4, \tag{47}$$

$$a_2=-(\overline{\gamma}^2+\overline{\beta}^2-e)/6, \tag{48}$$

$$a_3=-\overline{\gamma}e/2, \tag{49}$$

$$a_4=-e^2+\overline{\beta}^2. \tag{50}$$

The solution of the general equation of this type is given in [17]. Let r be any root of the quartic $Q(S_3)$, and if g_2 and g_3 are its two invariants, given by

$$g_2=a_0a_4-4a_1a_3+3a_2^2, \tag{51}$$

$$g_3=a_0a_2a_4+2a_1a_2a_3-a_2^3-a_0a_3^2-a_1^2a_4, \tag{52}$$

then the solution of Eq. (44) is

$$S_3=r+\frac{1}{4}\frac{Q'(r)}{\left[\wp(\overline{z};g_2,g_3)-\frac{1}{24}Q''(r)\right]}. \tag{53}$$

Here $\wp(\overline{z};g_2,g_3)$ is the standard Weierstrass elliptic function with invariants g_2 and g_3. The other two components of **S** can be found from this and the equation of motion.

This expression assumes a much simpler form if either period of the elliptic function $\wp(\overline{z})$ tends to infinity; this happens on any homoclinic orbit, and in particular, on the limit of these, the cusped orbit. In the former case, $(\overline{\beta},\overline{\gamma})$ lies inside the bifurcation curve (42). As the homoclinic orbit is approached, the real period of $\wp$ tends to infinity; the orbit may thus be found in terms of hyperbolic functions. Here, as we have seen, the quartic $Q(S_3)$ has a double root, so the equation of motion is now

$$\left(\frac{dS_3}{d\overline{z}}\right)^2=-\frac{1}{4}(S_3-r_0)^2(S_3-r_1)(S_3-r_2), \tag{54}$$

where r_0 is the double root at the saddle point. We can relate the roots (r_0,r_1,r_2) to $(\overline{\beta},\overline{\gamma})$ as follows. We set $r_0=\cos(\theta_0)$. This must be a root of both $Q(S_3)=0$ and $Q'(S_3)=0$. The first equation fixes the energy,

$$\frac{1}{2}\cos^2(\theta_0)-\overline{\gamma}\cos(\theta_0)+\overline{\beta}\sin\theta_0=e, \tag{55}$$

while the second relates θ_0 to $\overline{\beta}$ and $\overline{\gamma}$:

$$\frac{\overline{\beta}}{\sin(\theta_0)}+\frac{\overline{\gamma}}{\cos(\theta_0)}=1. \tag{56}$$

Now the other two roots are found directly,

$$r_1=2\overline{\gamma}-\cos(\theta_0)-\sqrt{\overline{\beta}\overline{\gamma}}\tan(\theta_0), \tag{57}$$

$$r_2=2\overline{\gamma}-\cos(\theta_0)+\sqrt{\overline{\beta}\overline{\gamma}}\tan(\theta_0). \tag{58}$$

There are two branches of the homoclinic orbit, originating from the saddle point r_0, and passing through either one of the simple roots r_1 or r_2. Equation (54) may be integrated in elementary functions, using the ansatz $S_3=r_1+1/u(\overline{z})$ to find the branch of the corresponding solution passing through $S_3=r_1$; this is

$$S_3=r_1+\frac{r_0-r_1}{1+\left(\frac{r_2-r_0}{r_2-r_1}\right)\operatorname{cosech}^2(k\overline{z})}, \tag{59}$$

where $k^2=(r_0-r_1)(r_2-r_0)/16$, which is positive. We note that $S_3\to r_0$ as $\overline{z}\to\pm\infty$, as required, while the remaining constant of integration is chosen so that if $\overline{z}=0$, $S_3=r_1$. Of course the other branch of the orbit may be found by exchanging r_1 and r_2 throughout. This solution could have been obtained directly by using the known limit of the Weierstrass $\wp$ function:

$$\wp\left(z;\frac{4}{3k^4},-\frac{8}{27k^6}\right)=k^2\left(\frac{1}{3}+\operatorname{cosech}^2(kz)\right). \tag{60}$$

As $(\overline{\beta},\overline{\gamma})$ approaches the curve (42), this solution degenerates further, for in this limit the imaginary period of the solution then tends to infinity as well; then the $\wp$ function reduces to just

$$\wp(z;0,0)=1/z^2, \tag{61}$$

and the solution becomes a rational function. Here the quartic $Q(S_3)$ now has a triple root, so the equation of motion is

$$\left(\frac{dS_3}{d\overline{z}}\right)^2=-\frac{1}{4}(S_3-r_0)^3(S_3-r_1), \tag{62}$$

with $r_0=\cos(\Theta)$ and $r_1=\cos(3\Theta)$. The orbit is then

$$S_3=r_1+\frac{r_0-r_1}{[1+16/((r_0-r_1)\overline{z})^2]}. \tag{63}$$

VI. SOME FINAL COMMENTS

The possible evolutions for **S** just described, and illustrated in the figures, are unstable on long time scales. This is because the perturbing terms $\mathcal{H}_I$, which are responsible for the change in polarization state **S**, also cause the soliton pulse to shed radiation as it propagates along the fiber. Put simply, if the perturbing terms $\mathcal{H}_I$ are of order ϵ, then the energy of the radiation generated will be of order ϵ^2. The net effect of this process on the evolution of **S** that we have described is a slow drift to lower energy levels e. This is essentially the origin of the fast mode instability [18]. This effect is only absent at stationary points of the polarization dynamics, so stable equilibrium is only reached at the (unique) minimum of H, when $\partial H/\partial \mathbf{S}$ and **S** are antiparallel; this corresponds to the slow mode. More drastic effects occur if nonautonomous perturbations are considered. It is often the case that γ varies randomly with distance $\overline{z}$ down the fiber. Any such variation that makes a Hamiltonian system nonautonomous can ''break'' a homoclinic orbit, leading to a chaotic evolution for **S**. We demonstrate this for the simpler case where γ is taken to be $\gamma(\overline{z})=\epsilon\cos(\Omega\overline{z})$, corresponding to ''rocking'' of the birefringence axes of the fiber. Menyuk and Wai [19] discuss the field evolution of fibers with rocked birefringence axes. They point out that in the linear case, that is, with the parameter $\overline{B}=0$, the trajectory on the Poincaré sphere describing the polarization state of the optical field is quasiperiodic. The motion combines the frequencies $\beta+\epsilon^2\Omega^2/(16\beta)$ and Ω. Almost always, the frequencies are incommensurate and the trajectory will then fill some region of the Poincaré sphere ergodically. If, however, they are commensurate, the trajectory executes a Lissajous figure on the sphere.

We can get an idea of the effect of small perturbations on the homoclinic orbit, for instance in the case $\overline{\beta},\overline{B}\neq 0$, and $\overline{\gamma}=\epsilon\cos(\Omega\overline{z})$, where $0<\epsilon\ll 1$, by applying Mel'nikov analysis to the system. This requires calculation of the function $M(\overline{z_0})$ defined below; it is known that if $M(\overline{z_0})$ has simple zeros, the effect of the perturbation is to break the homoclinic connection, leading to infinitely many transversal intersections between the stable and unstable manifolds of the saddle point, giving rise to horseshoes and chaotic evolution. Otherwise, if $M(\overline{z_0})$ is bounded away from zero, the unstable and stable manifolds will not intersect at all. If the evolution equation is given in the form

$$\frac{d\mathbf{S}}{d\overline{z}}=\mathbf{S}\times\frac{\partial H}{\partial \mathbf{S}}+\epsilon\mathbf{g}(\mathbf{S},\overline{z}), \tag{64}$$

the Mel'nikov function for an orbit is defined [20] as a functional of $\mathbf{S}(\overline{z})$:

$$M(\overline{z_0})=\int_{-\infty}^{\infty}\frac{\partial H}{\partial \mathbf{S}}(\overline{z})\cdot\mathbf{g}(\mathbf{S},\overline{z}+\overline{z_0})d\overline{z}; \tag{65}$$

here it is evaluated on the homoclinic solution $\mathbf{S}^{(0)}(\overline{z})$. Thus $M(\overline{z_0})$ can be thought of as a measure of the first-order change in the Hamiltonian H along the orbit as a result of the perturbation **g**. Using the equation of motion, and integrating by parts, we reduce the integral to

$$\begin{aligned}\int_{-\infty}^{\infty}\overline{\beta}\cos(\Omega(\overline{z}+\overline{z_0}))S_2^{(0)}d\overline{z}\\ =-\int_{-\infty}^{\infty}\sin(\Omega(\overline{z}+\overline{z_0}))\Omega(S_3^{(0)}-r_0)d\overline{z}\\ =-\sin(\Omega(\overline{z_0}))\int_{-\infty}^{\infty}\cos(\Omega(\overline{z}))(S_3^{(0)}-r_0)d\overline{z},\end{aligned} \tag{66}$$

where r_0 is the value of S_3 at the saddle point, equal to 1 in this case. Since the integral in the last line does not vanish identically, we see that $M(\overline{z_0})$ has infinitely many simple zeros, corresponding to infinitely many transversal intersections between the stable and unstable manifolds, resulting in chaotic dynamics. There is no threshold for the onset of this chaotic behavior; this is what we would expect in the absence of dissipation. This result extends immediately to the case

$$\overline{\gamma}=\overline{\gamma_0}+\epsilon\cos(\Omega\overline{z}), \tag{67}$$

while an extension to the case with random fluctuations in $\overline{\gamma}$ is also possible in principle.

We have seen that a careful consideration of the Hamiltonian structure and symmetries of Eq. (1) leads to a self-consistent description of the polarization dynamics. This permits a complete description of the bifurcations of the reduced system, and an exact calculation of its orbits, in particular the family of homoclinic orbits emanating from saddle points. The extension to nonautonomous systems of the same type is also possible; here we expect perturbations in, for instance, the twist of the fiber, to lead to chaotic motion in the neighborhood of the homoclinic orbit.

ACKNOWLEDGMENT

One of the authors (S.M.B.) is pleased to acknowledge financial support from the EPSRC.

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